

Recovering Probabilities and Risk Aversion from Options Prices and Realized Returns

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Abstract

This paper summarizes a program of research we have conducted over the past four years. So far, it has produced two published articles, one forthcoming paper, one working paper currently under review at a journal, and three working papers in progress. The research concerns the recovery of market-wide risk-neutral probabilities and risk aversion from option prices.

The work is built on the idea that risk-neutral probabilities (or their related state-contingent prices) are an amalgam of consensus subjective probabilities and consensus risk aversion. The most significant departure of this work is that it reverses the normal direction of previous research, which typically moves from assumptions governing subjective probabilities and risk aversion, to conclusions about risk-neutral probabilities. Our work is partially motivated by the conspicuous failure of the Black-Scholes formula to explain the prices of index options in the post-1987 crash market.

First, we independently estimate risk-neutral probabilities, taking advantage of the now highly liquid index option market. We show that, if the options market is free of general arbitrage opportunities and we can at least initially ignore trading costs, there are quite robust methods for recovering these probabilities.

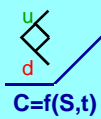
Second, with these probabilities in hand, we use the method of implied binomial trees to recover a consistent stochastic process followed by the underlying asset price.

Third, we provide an empirical test of implied trees against alternative option pricing models (including “naïve trader” approaches) by using them to forecast future option smiles.

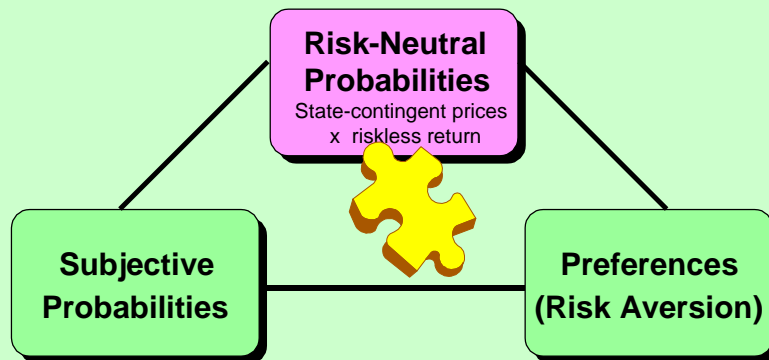
Fourth, we argue that realized historical distributions will be a reliable proxy for certain aspects of the true subjective distributions. We then use these observed aspects together with the option-implied risk-neutral probabilities to estimate market-wide risk aversion.

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Risk-Neutral Probabilities: The Link between Probabilities and Preferences



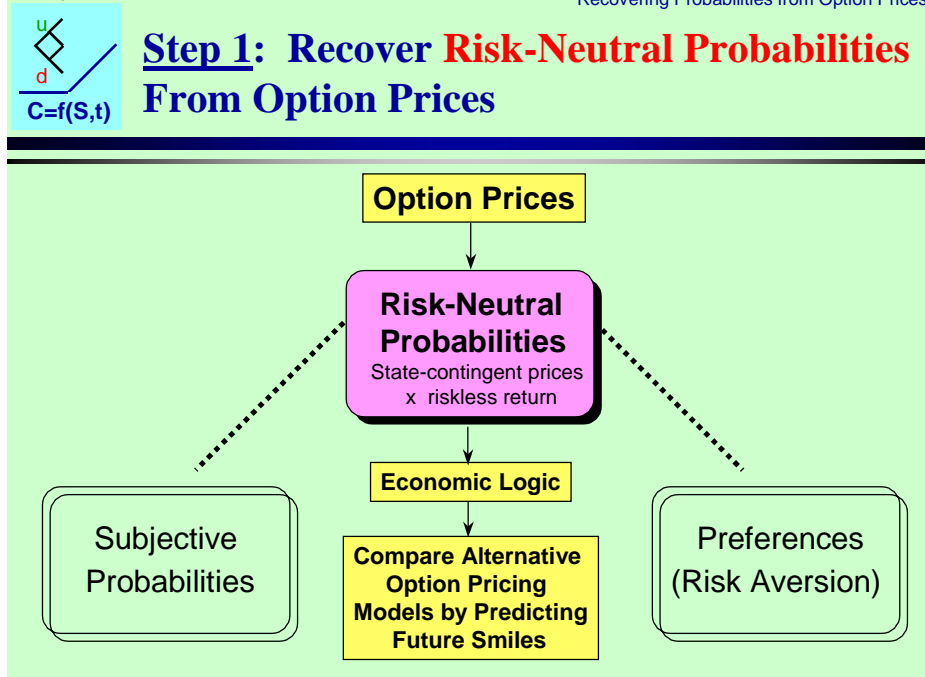
$$\text{Subjective Probabilities} = \text{Risk-Neutral Probabilities} \times \text{Risk Aversion Adj}$$

I. Introduction

Standard equilibrium models in financial economics are, in their essential nature, ways of determining **state-contingent prices**: the price today of a dollar to be received at only a specific date in the future and given a specific description of the state-of-the-economy at that time. If there are no **riskless arbitrage opportunities**, each of these prices is positive. The sum of the state-contingent prices for dollars received at a single date over all possible states is the current price of a dollar received for sure at that date. This is one divided by the current **riskless return** for that date. Therefore, multiplying the state-contingent prices by this return converts them into a probability measure over the states, which financial economists call **risk-neutral probabilities**. *This paper is largely about ways of recovering these probabilities from the current riskless return, the currently observed prices of traded assets, and the current prices of traded derivatives on those assets.*

The usual way of applying the equilibrium model goes about this differently. It takes as given the subjective probabilities and risk-preferences of an “average investor” and uses these to determine the risk-neutral probabilities. The argument is that, *ceteris paribus*, a risk-neutral probability will be higher the higher the **subjective probability** of achieving its associated state: the probability measuring the investor’s degree of belief that the corresponding state will occur. If the investor were indifferent to risk, then corresponding risk-neutral and subjective probabilities would be equal. However, the investor may value an extra dollar more highly in one state than another. From example, if he was risk averse, he would value an extra dollar more highly in states when, *ceteris paribus*, his wealth were relatively low. This motivates him to spread his wealth out evenly across states. However, aggregate economic uncertainty prevents this since the aggregate supply of dollars in all states is not the same. As a result, what he is willing to pay today for a dollar received tomorrow not only depends on his subjective probabilities but also on his degree of **risk aversion**. Risk-neutral probabilities, therefore, can be interpreted as subjective probabilities which are adjusted upward (downward) if they correspond to states in which dollars are more (less) highly valued.

In the standard approach, given the riskless return and having determined the state-contingent prices in this way, assuming **perfect markets**, traded securities are simply portfolios of state-contingent securities. Therefore, the value of traded securities can be easily calculated; and the model may be tested by comparing these values to quoted market prices. As a practical matter, the standard equilibrium model has been difficult to test empirically because it has been difficult to identify the relevant subjective probabilities and risk-aversion.

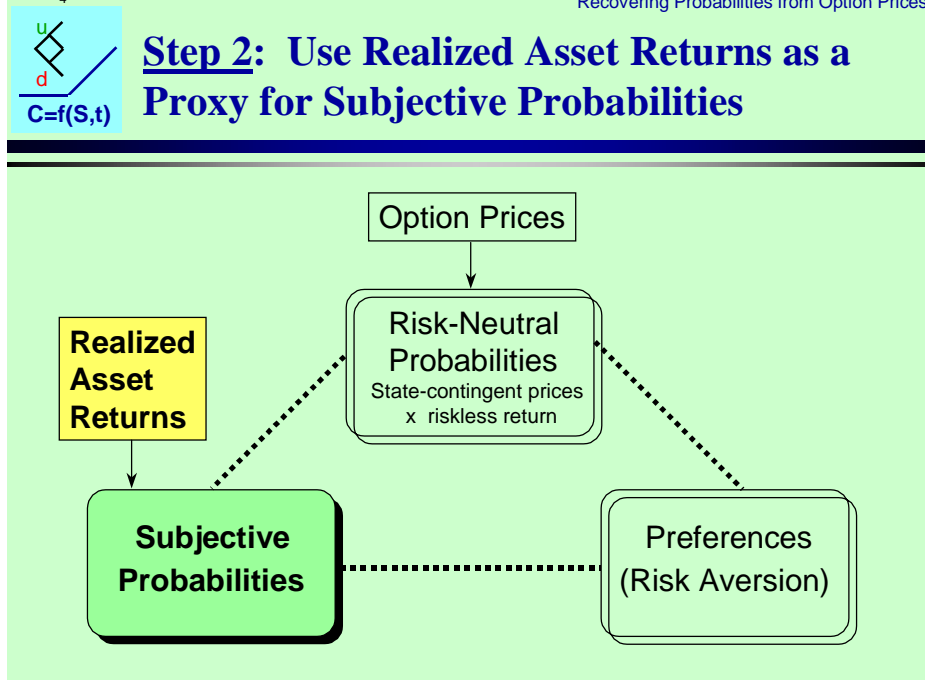


The approach of this paper is to break this Gordian knot and determine the risk-neutral probabilities directly — and only then try to say something about how these probabilities decompose into subjective probabilities and risk-aversion.

We take as datum the current prices of traded options on a proxy for the **market portfolio**: the portfolio of assets which has the same proportionate payoffs across states as aggregate wealth. Our proxy is the portfolio measured by the Standard and Poor's 500 Index of common stocks (**S&P 500 Index**). Since a highly liquid market has existed for about a decade on a wide variety of different European puts and calls on the S&P 500 Index, it is tempting to take advantage of this comparatively recent development in financial markets. Admittedly, this is an incomplete and probably biased proxy and some, though not all, of our results may be affected by this.

We begin by discussing methods of recovering risk-neutral probabilities from the concurrent prices of these options (along with the concurrent level of the index and the riskless return). If these prices were set according to the **Black-Scholes formula**, our task would be a simple one [Black and Scholes 1973]. In that case, the entire risk-neutral probability distribution could be summarized by its volatility (its mean must equal the riskless return). Unfortunately, since the stock market crash of 1987, the Black-Scholes formula fits the market prices of S&P 500 Index options very poorly. So we need to investigate other methods of recovering these probabilities from market prices. If European options expiring on the target expiration date existed on the Index spanning all possible striking prices from zero to infinity, then (ignoring trading costs) the simultaneously observed prices of these options would uniquely determine the risk-neutral probability distribution [Breen and Litzenberger 1978]

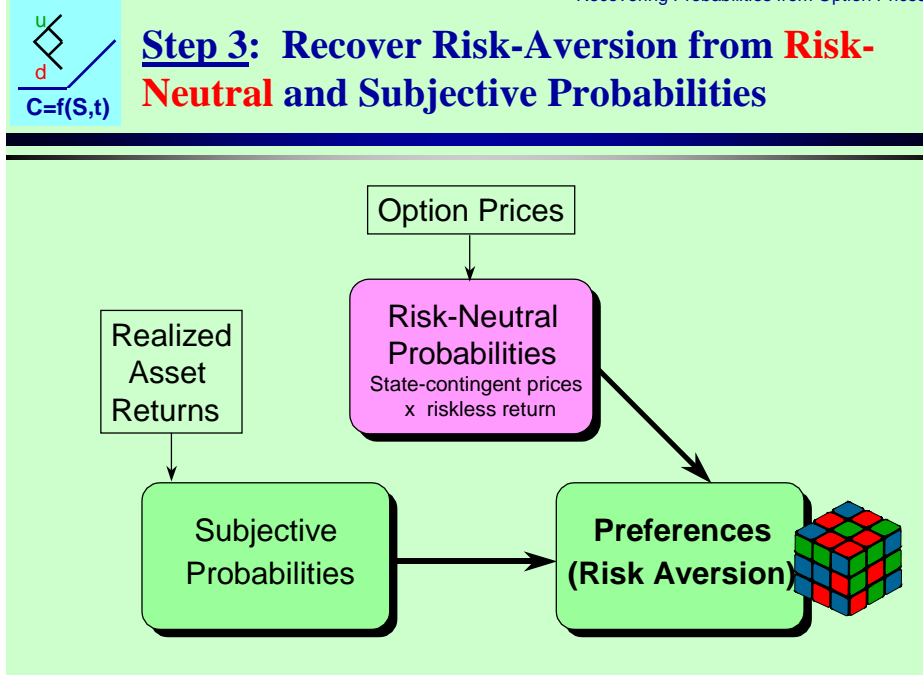
Of course, such a complete set of options does not currently exist. In practice, striking prices are set at discrete intervals, and there is a lowest (highest) striking price significantly greater (less) than zero (infinity). This opens the recovery problem to different possible methodologies. We consider a number of possibilities, including quadratic optimization and the method of maximizing smoothness [Jackwerth and Rubinstein 1996]. Because of the richness of market for S&P 500 Index options, the most important properties of the recovered distribution are not sensitive to the particular methodology: its extreme leptokurtosis (peakedness) and left skewness. While all tested methods result in much more probability in the lower left tail than the lognormal, because there are few options with striking prices covering that region, the exact distribution of this greater probability in this region is sensitive to the methodology chosen. For example, whether the distribution contains another mode in this region can depend on the recovery methodology.



We might hope to recover even more information from option prices, in particular, the stochastic process followed by the underlying index price. Unfortunately, given the recovered risk-neutral probability distribution for a given expiration date, there are an infinite number of possible stochastic processes which are consistent with these prices. To sort through these processes, we need to make additional assumptions. We design a model that is as close as possible to the standard binomial option pricing model while allowing options to be valued under an arbitrary prespecified risk-neutral probability distribution applying to a single given expiration date which corresponds to the ending nodes of a recombining binomial tree. We call the resulting stochastic process an **implied binomial tree** [Rubinstein 1994]. The strongest assumption we make initially (also a property of standard binomial trees) is that all paths leading to the same ending node have the same risk-neutral probability. Applying this model to post-crash option prices produces a tree of local (or one move) volatilities with the following general features:

- n on a given date prior to expiration, local volatilities are higher the lower the level of the underlying index;
- n for a given change from the initial underlying index price, the faster it occurs, the greater the change in the local volatility;
- n for index levels near the initial underlying index price, the farther into the future the local volatility, the lower it tends to be.
- n One line of research has been to drop the assumption that all paths leading to the same ending node have the same risk-neutral probability. Fortunately, the model can be generalized by adding path weighting parameters which can be calibrated so that the generalized implied binomial tree now also fits the prices of options which expire on earlier dates [Jackwerth 1997].

Stepping away from the purely modeling problems, we ask what fundamental features of the economy could create the recovered risk-neutral distribution and implied binomial tree. We provide four potential explanations. A goal of future research will be to find some way of determine what combination of these explanations actually underlies the observed phenomena. With this in hand, we will have a much deeper understanding than we now have of the economic forces that determine security prices; and we will be able to anticipate the effects on security prices of structural economic changes.

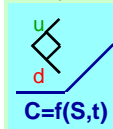


While the recovered risk-neutral probability distribution for a given expiration date is quite robust to our assumptions, this is not true for the implied binomial tree (which requires a much stronger set of assumptions). Fortunately, an implied tree has several empirical implications which are amenable to empirical tests [Jackwerth and Rubinstein, working paper in progress]. Most important of these is the prediction of future Black-Scholes **implied volatility smiles** given the corresponding future underlying index price. Since other option pricing models also can be interpreted as making this kind of forecast, we have an opportunity not only to test the validity of implied binomial trees, but also to compare its predictive power to that of other popular option pricing models or cruder smile prediction techniques used in practice (“naïve trader” models). We find, that despite the greater sophistication of “academic” approaches, a very crude rule-of-thumb used in practice, produces the best predictions in our post-crash empirical sample. However, while as expected the Black-Scholes formula does very poorly, a **CEV model** and implied binomial trees only do a little worse than the best naïve trader model.

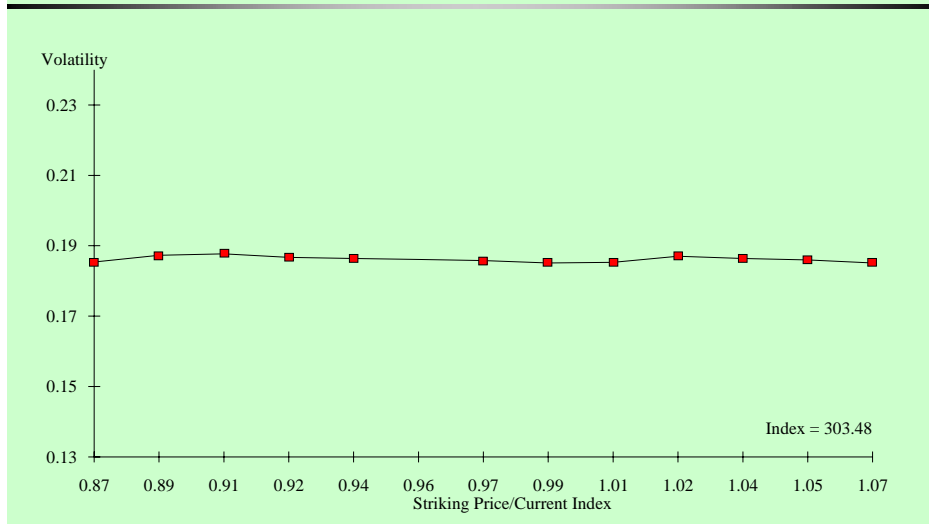
Relying only on our robust approach to estimate expiration-date risk-neutral distributions, we then try to break these risk-neutral probabilities apart into a product of subjective probabilities and risk aversion [Jackwerth 2000]. We measure subjective probabilities using the traditional technique of historical frequency distributions. In the past, the two key problems with this kind of inference have been first estimating the mean of the subjective probability distribution (since the mean of the realized frequency distribution is highly unstable), and second the difficulty of ascertaining the shape of the tails. Fortunately, we show that our conclusions about inferred market-wide risk aversion need rely only on information about the shape of the subjective distribution near its mean, wherever that mean may be.

Unfortunately, the logic of the model breaks down, implying for example that in aggregate the market actually prefers risk, or at best has increasing absolute risk aversion. We then consider a number of explanations for this implausible result. The most disturbing of these is that the index options market is highly inefficient. We test this hypothesis by following a post-crash investment strategy where we accumulate profits by rolling over a sequence of out-of-the-money puts and find that this strategy leads to highly excessive risk-adjusted excess returns even if we adopt general risk adjustments which account for the utility benefits of positive skewness and even if we inject frequent crashes of the October, 1987 magnitude into the historically realized index returns.

The research reported in this paper summarizes a four-year effort, some published and some still in unfinished working paper form. To pursue this in more detail, it will be necessary to look at those papers.



Typical 164-Day Pre-Crash Smile Implied combined volatilities of S&P 500 Index options July 1, 1987 (8:59 am)



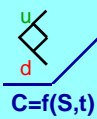
II. The Problem

The interest in this research arises because the popular approach of explaining option prices – the Black-Scholes formula – fails miserably to explain post-crash US index option prices (as well as post-crash index option prices in several other countries). This anomaly stands out since the formula works much better in explaining the prices of most individual stock and foreign currency options.

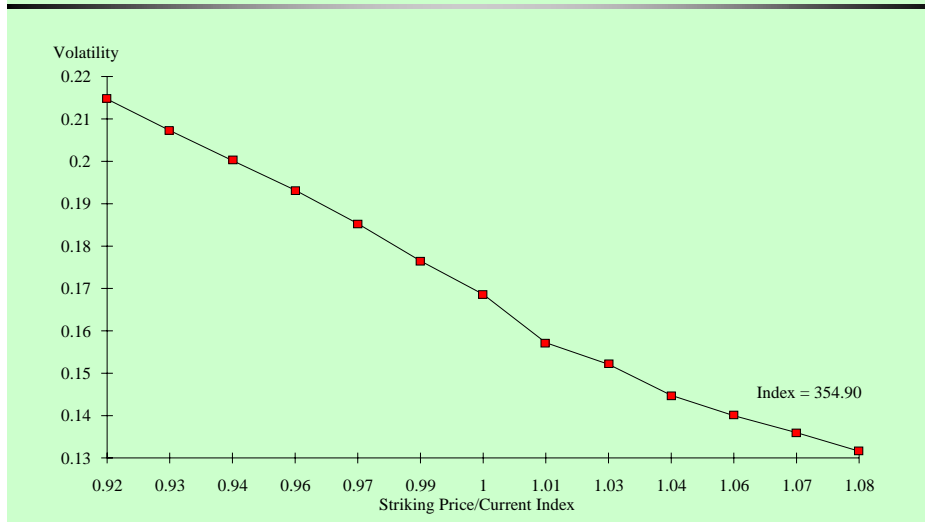
The attached graph shows the implied volatility smile for 164-day S&P 500 Index options traded on the Chicago Board Options Exchange on July 1, 1987 at 8:59 a.m. Central Time. If the Black-Scholes formula were true for these options, the smile should be perfectly flat. There can only be one risk-neutral probability distribution for the underlying index behind these options (since all the options are on the same underlying index and are only exercisable on the same date). Black-Scholes assume that this distribution is lognormal, with its two free parameters, mean and variance, fully determined by the riskless return and implied volatility.

As seen in the graph, the smile is remarkably flat, well within the bounds of realistic trading costs. So in this pre-crash period, the Black-Scholes formula appears to be doing extremely well, justifying its reputation as the last word in option pricing.* Moreover, this time can be shown to be typical for these options priced before the 1987 stock market crash.

* Note, however, that a flat smile is only a necessary condition for the Black-Scholes formula to hold, not also a sufficient condition.



Typical 164-Day Post-Crash Smile Implied combined volatilities of S&P 500 Index options January 2, 1990 (11:07 am)



In stark contrast, after the stock market crash a very steep smile developed in the S&P 500 Index option market, roughly similar to the above graph from mid-1988 to the present. This smile betrays an extreme departure from the predictions of the Black-Scholes formula. One way to place a lower bound on this departure is to select as the implied volatility in the Black-Scholes formula the volatility that minimizes the largest dollar or percentage error of a single option price over the set of all available options. This gives Black-Scholes the full benefit of the doubt. However, even if we do this, one of the options will have a pricing error of about \$4.00, or one will have a pricing error of 15%. Such errors are probably well beyond the range that could be created by realistic trading costs. In any event, it is difficult to believe that changes in trading costs could account for the change in the smile across the divide of the stock market crash.

Options with shorter time-to-expiration are more liquid. Had we chosen these, the smile would have even been steeper, implying even greater departure from Black-Scholes predictions than for the 164-day options above.

This pricing deviation from Black-Scholes is striking for several reasons:

- n it has existed more or less continuously over a 10-year period;
- n it resides in one of the most liquid and active option markets with a very large open interest;
- n it is found in a market which, one might argue on theoretical grounds, is most likely to be the one for which the Black-Scholes formula works best.*

This situation just cries out for an alternative way to approach option pricing.

* It is sometimes argued that while the Black-Scholes formula can be expected to hold for individual equity options since their underlying asset returns should be approximately lognormal, it will not hold for index options whose underlying index would then be a weighted sum of lognormal variables, clearly itself not lognormal. However, we believe this puts the cart before the horse. It seems to us more probable that when "God" created the financial universe, he made the market portfolio lognormal; and man, in his efforts to create exchange arrangements, then created individual equities and other securities he called bonds with returns which are not lognormal. We suspect that empirical analysis would show that the returns of diversified portfolios of stocks are closer to lognormal than their typical constituent components. Moreover, jumps, the Achilles heal of the Black-Scholes formula, are much more likely to prove a problem for a typical individual equity than for an equity index.



Recovering Risk-Neutral Probabilities: Optimization Method

$$\min \sum_j (\mathbf{P}_j - \mathbf{P}'_j)^2 \text{ subject to:}$$

$$\sum_j \mathbf{P}_j = 1 \text{ and } \mathbf{P}_j \geq 0 \text{ for } j = 0, \dots, n$$

$$\mathbf{S}^b \leq S \leq \mathbf{S}^a \text{ where } S = (d^t \sum_j \mathbf{P}_j \mathbf{S}_j) / r^t$$

$$\mathbf{C}_i^b \leq C_i \leq \mathbf{C}_i^a \text{ where } C_i = (\sum_j \mathbf{P}_j \max[0, S_j - \mathbf{K}_i]) / r^t \text{ for } i = 1, \dots, m$$

where j indexes the ending binomial nodes from lowest to highest

\mathbf{P}_j \equiv implied (**posterior**) ending nodal risk-neutral probabilities

\mathbf{P}'_j \equiv prespecified (**prior**) ending nodal lognormal risk-neutral probabilities

\mathbf{S}_j \equiv underlying (ex-payout) asset prices at end of standard binomial tree

\mathbf{S}^b (\mathbf{S}^a) \equiv current **observed** bid (ask) underlying asset price

\mathbf{C}_i^b (\mathbf{C}_i^a) \equiv current **observed** bid (ask) call option price with striking price \mathbf{K}_i

d \equiv **observed** annualized payout return

r \equiv **observed** annualized riskless return

t \equiv time to expiration

III. Recovering Risk-Neutral Probability Distributions

One possibility is to let the option prices speak for themselves. In contrast with Black-Scholes, the approach advocated here is non-parametric in the sense that any risk-neutral probability distribution could result. Instead, Black-Scholes begin by assuming that the risk-neutral distribution must be lognormal; the only question remaining is what is its volatility (its mean is anchored to the riskless return).

However, whatever methodology is selected should satisfy the following properties:

n as the number of available options with different striking prices becomes denser, or spans a larger range, the methodology should result in a recovered distribution which is closer in a useful sense to the unique distribution recovered from a complete set of options

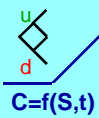
n if the methodology uses a **prior distribution** as input and if the option prices can be explained by this distribution, the **recovered (posterior) distribution** should be the same.

n if any buy-and-hold arbitrage opportunities exist among the options, the underlying asset, and cash, the methodology should fail to recover any distribution.

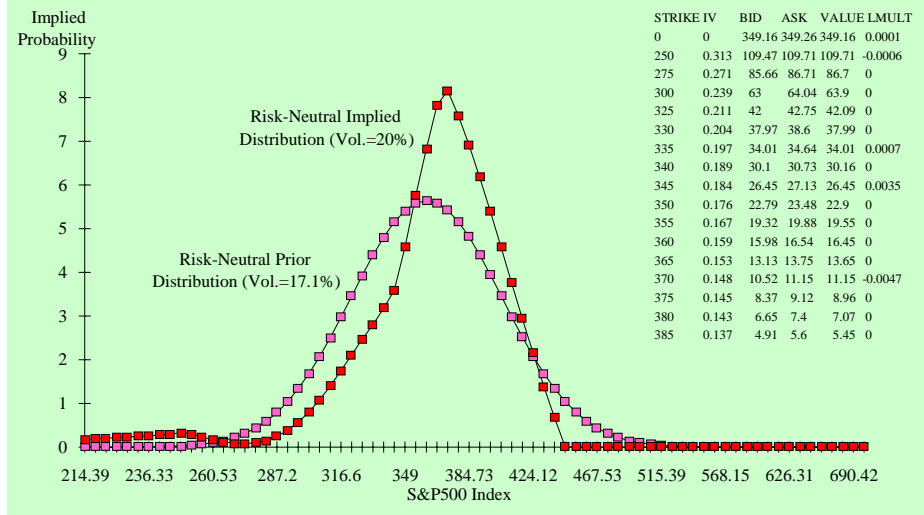
n if option prices were determined by the Black-Scholes formula, the recovered distribution should be lognormal.

In the attached picture, we start by making a prior guess of the implied risk-neutral distribution, \mathbf{P}'_j , over all possible levels of the underlying asset price \mathbf{S}_j at expiration, $j = 0, 1, 2, \dots, n$. Also assumed known are the current bid and ask underlying asset prices, \mathbf{S}^b and \mathbf{S}^a , the current bid and ask prices of associated call options, \mathbf{C}_i^b and \mathbf{C}_i^a , with striking prices \mathbf{K}_i , $i = 1, 2, 3, \dots, m$, all with the same time-to-expiration, t , the current annualized return on a riskless zero-coupon bond maturing on the expiration date, r , and the current annualized payout return on the underlying asset through the expiration date, d .

The problem is to determine from this information the posterior risk-neutral probabilities \mathbf{P}_j , which explain the current prices of the options as well as the underlying asset. The first constraints in the attached picture, $\sum_j \mathbf{P}_j = 1$ and $\mathbf{P}_j > 0$ assure that the \mathbf{P}_j will indeed be probabilities. The second constraints, $\mathbf{S}^b \leq S \leq \mathbf{S}^a$ and $S = d^t \sum_j \mathbf{P}_j \mathbf{S}_j / r^t$, assure that the current value placed on the underlying asset, S , is the discounted expected value of its future possible prices using the posterior risk-neutral probabilities after adjusting for payouts, and that this value lies between the market bid and ask prices.



Prior and Implied Risk-Neutral 164-Day Probabilities S&P500 Index Options: January 2, 1990 (11:00 am)



The third constraints, $C_i^b \leq C_i \leq C_i^a$ and $C_i = \sum_j P_j \max[0, S_j - K_i]/r_t$, assure that the current value placed on the calls, C_i , is the discounted expected value of its possible future payoffs using the posterior risk-neutral probabilities, and that this value lies between the market bid and ask prices.

Among all the posterior risk-neutral probability distributions which satisfy these constraints, the distribution chosen by this methodology is the one that is "closest" to the prior distribution in the sense of minimizing the average squared distance between these two probability distributions.

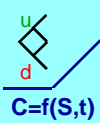
While there is some arbitrariness created by the assumed prior distribution and the quadratic measure of closeness, the method does satisfy the previous four properties claimed to be desirable for any technique for recovering risk-neutral probabilities from options.*

The attached picture shows a typical post-crash recovered distribution by this method. The distribution is based on the simultaneously observed bid-ask prices of 16 164-day European S&P 500 Index options with striking prices ranging from 250 to 385 and a current index level of 349.16 on January 2, 1990. This information closely matches the post-crash smile reported earlier.

The lighter-colored distribution is the one we would expect from Black-Scholes using the at-the-money options to determine the single implied volatility (17.1%) applied to all the options. It is derived by taking logarithms of returns to be a normal distribution. In contrast, the darker-colored distribution is the recovered posterior distribution. Even though this distribution was in sense prejudiced to come up lognormal (since the prior was lognormal), its shape is markedly different, showing significant left skewness, much higher leptokurtosis and slight bimodality. Perhaps the key feature is the much larger concentration of probability in the lower left-hand tail.

While we don't present the detailed evidence here, it turns out that these features of the recovered distribution are continuously displayed from about mid-1988 to the present in this market. On the other hand, prior to October 1987, the two distributions are nearly indistinguishable. The crash, then, marks a divide in the pricing of S&P 500 Index options. Evidence now available on smiles for other US index options and for options on foreign stock market indexes is confirmatory: the features observed here for risk-neutral distributions carry over to other equity index options [Gemmill and Kamiyama (1997)].

* The fourth property, recovering a lognormal distribution if all available options have the same Black-Scholes implied volatility, is met if the prior distribution is assumed to be lognormal -- as assumed in the attached graph.



Recovering Risk-Neutral Probabilities: Alternative Nonparametric Methods

- **Basic method**
- **Smile interpolation method**
- **Optimization methods:**
 - **Quadratic:** $\sum_j (\mathbf{P}_j - \mathbf{P}_j')^2$
 - **Goodness of fit:** $\sum_j (\mathbf{P}_j - \mathbf{P}_j')^2 / \mathbf{P}_j'$
 - **Absolute difference:** $\sum_j |\mathbf{P}_j - \mathbf{P}_j'|$
 - **Maximum entropy:** $-\sum_j \mathbf{P}_j \log(\mathbf{P}_j / \mathbf{P}_j')$
 - **Smoothness:** $\sum_j (\mathbf{P}_{j-1} - 2\mathbf{P}_j + \mathbf{P}_{j+1})^2$ [NO PRIOR]

With enough options, the methodology we have used for recovering probabilities becomes insensitive to choice of prior or our choice of the quadratic measure of closeness. In effect, the recovered distribution becomes driven solely by the constraints.

To test the robustness of the approach with the number and span of options usually available for S&P 500 Index options, we tried alternative optimization criteria besides the quadratic. Alternative criteria which could replace $\min \sum_j (\mathbf{P}_j - \mathbf{P}_j')^2$ include:

- n goodness of fit: $\min \sum_j (\mathbf{P}_j - \mathbf{P}_j')^2 / \mathbf{P}_j'$
- n absolute difference: $\min \sum_j |\mathbf{P}_j - \mathbf{P}_j'|$
- n maximum entropy: $\min -\sum_j \mathbf{P}_j' \log(\mathbf{P}_j / \mathbf{P}_j')$
- n maximum smoothness: $\min \sum_j (\mathbf{P}_{j-1} - 2\mathbf{P}_j + \mathbf{P}_{j+1})^2$

Each of these has its own rationale. The goodness of fit criterion places greater weight on states with lower probabilities; the absolute difference criterion places less weight on the most extreme differences between priors and posteriors. Perhaps, from a purely theoretical standpoint, the maximum entropy criterion is superior since it selects the posterior that has the highest probability of being correct given the prior. The maximum smoothness criterion, similar to fitting a cubic spline, minimizes the sum of the square of the second derivative $\partial^2 P / \partial S^2$ over the entire probability distribution. The expression $\mathbf{P}_{j-1} - 2\mathbf{P}_j + \mathbf{P}_{j+1}$ is a finite difference approximation for this second derivative. Note that this last criterion does not rely on a prior.

In practice, although the maximum entropy criterion may be best in theory, it is difficult to apply. In contrast, using the maximum smoothness criterion almost permits the problem to be transformed into solving a set of triangular linear equations, and so produces very quick and reliable solutions. In any event, in the region between the lowest and highest striking prices, all the optimization criteria result in almost the same recovered probability distribution. In each case, the distribution is also heavily skewed to the left (post-crash). However, while all approaches agree that the recovered distribution has much more probability in the lower left tail than the normal (post-crash), they disagree about how that probability is distributed in that tail. For example, one approach may produce slight bimodality while another may not.

For those methods that require priors, again it turns out that, at least for S&P 500 Index options, available striking prices are sufficiently dense that the implied risk-neutral distribution is not particularly sensitive to the imposition of a uniform in place of a lognormal prior



Implied Binomial Trees: Assumptions

Objective: value options for arbitrary risk-neutral expiration date probability distributions

- ① Underlying asset follows binomial process
- ② Binomial tree is recombining
- ③ Ending nodal values ordered from lowest to highest
- ④ Riskless (and payout) return constant
- ⑤ All paths leading to the same ending node have the same risk-neutral probability

New Objective: generalize fifth assumption but retain the simplicity of the recursive solution process

IV. Recovering Risk-Neutral Stochastic Processes

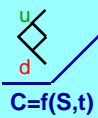
As we indicated earlier, obtaining a good estimate of the risk-neutral probability distribution at the expiration date is only part of the story. We also want to recover the stochastic process that leads to this distribution. In a discrete version of the Black-Scholes model, this can be described by a recombining binomial tree with constant multiplicative up and down moves, and constant riskless and payout returns. After a sequence of these moves, the probabilities at the end of the tree can be made to approximate closely a risk-neutral lognormal distribution with a prespecified volatility and mean [Cox, Ross and Rubinstein 1979]. However, if the target risk-neutral distribution departs significantly from lognormal, as we have seen for the post-crash index option market, this simple binomial stochastic process must perform be inconsistent with this.

So one might ask, is there a way to modify the binomial model which leaves its major advantages intact -- its intuitive simplicity and numerical tractability -- but at the same time is consistent with the actual recovered risk-neutral distribution? It turns out this can be done even while retaining the main attractive features of the binomial approach:

- n binomial price moves,
- n recombining nodes,
- n ending nodal values organized from lowest to highest,
- n constant riskless and payout returns, and
- n all paths leading to the same ending node having the same risk-neutral probability.

This last feature means that if you stand at a node at the end of the tree and look backwards, you will see many paths from the beginning of the tree that lead to that node. Each of these paths has the same probability. This does not mean that all paths in the tree have the same probability, but that *conditional on ending up at a particular terminal node*, the paths have the same probability.

However, in an important way the modified binomial tree differs from the standard tree: *it does not require constant move sizes*. It allows the local volatility of the underlying asset return to vary with changes in the underlying asset price and time. In addition, it can be shown that given the ending risk-neutral distribution, the riskless and payout returns, and with the above assumptions, there is a unique consistent binomial tree, which moreover, preserves the property that there are no arbitrage opportunities in the interior of the implied tree (all risk-neutral move probabilities, although they may be different at each node, are non-negative).



Annualized Global Volatility Structure

January 2, 1990: 10:00 am

S&P 500 Index	Days into the Future						
	0	12	32	47	61	76	91
397			9.3	9.9	10.4	11.1	11.9
386			10.8	11.1	11.8	12.4	13.0
376		11.8	12.1	12.5	12.9	13.4	13.9
365		14.3	14.0	14.0	14.1	14.5	14.8
355*	20.0	18.8	17.1	16.2	15.8	15.7	15.9
338		26.6	24.1	22.1	20.4	18.8	17.8
318		34.0	31.5	29.5	27.5	25.1	22.5
297		38.6	36.4	34.4	32.6	30.3	28.2
273			38.7	36.7	34.5	32.2	30.2

*S&P 500 Index reported at 10:00 am is 354.75.

Based on S&P 500 Index June Call and Put Options Maturing in 164 Days

Perhaps the most undesirable feature of this modified binomial approach -- even though it is shared with the standard binomial approach -- is the assumption that all paths leading to the same ending node have the same risk-neutral probability. Fortunately, it can be shown that this last assumption can be dropped, and the implied tree can be tractably designed to fit simultaneously options on the same underlying asset but with different times-to-expiration as well as different striking prices.

The modified binomial approach can be used to imply the stochastic process for S&P 500 Index options on January 2, 1990 with 164 days-to-expiration. Instead of depicting the resulting process in the usual way as a tree of up and down moves, it is perhaps more instructive to depict the tree in terms of the evolution of implied volatility, as in the attached picture. The volatility shown here is actually the annualized volatility from its associated node to the end of the tree, called the "global volatility." It turns out that this will be similar to the Black-Scholes implied volatility for an option which is at-the-money at that node.

As we can see from the tree, the global volatility starts at 20% on January 2 when the options have 164 days-to-expiration. If the Index falls 16% to 297 over the next twelve days (so the options now have only 152 days-to-expiration), the volatility almost doubles to 38.6%. This may seem like an excessive increase in volatility, but something like this happened during the 1987 stock market crash. If this same fall were to take 91 days, then the volatility would only rise to 28.6%. On the other hand, if the Index rises, the volatility falls. Also note that if the Index ends up in 91 days at the same level as it started at 355, then the volatility will fall to 15.9%. The implied binomial tree shows that one way to make sense out of the downward sloping smile (or alternatively, the left skewness of the recovered probability distribution) in index options is to suppose that the implied volatility varies inversely with the underlying asset price.

It is important to realize that these predictions concerning volatility are all recovered from the January 2, 1990 prices of S&P 500 Index options. They embody predictions about future option prices and are therefore amenable to an empirical test. For example, if the predictions are accurate, when we move 12 days into the future, say to January 14, 1990, if the Index is then at 297, at-the-money options should be priced in the market such that their implied volatility is about 38.6%. Of course, the world is much more complex than our model, but we still might hope that the model gives an unbiased, low error variance prediction of future implied volatility, conditional on the future underlying asset price and the time remaining to expiration. One of our tasks will be to check this out and to compare the predictions from this method of constructing implied binomial trees to the predictions from other approaches.



Forecasting Future ATM Implied Volatility

σ_t \equiv current ATM implied volatility

σ_{t+14} \equiv future ATM implied volatility (14 days later)

S&P 500 Index Options

Pre-crash 86/04/02-87/09/04:

$$\sigma_{t+14} = a\sigma_t: \quad r^2 = .47, \quad a = 1.0024$$

$$\sigma_{t+14} - \sigma_t = (b/n)\log(S_{t+14}/S_t): \quad r^2 = .04, \quad b = -.6518$$

$$\sigma_{t+14} = a\sigma_t + (b/n)\log(S_{t+14}/S_t): \quad r^2 = .49, \quad a = 1.0058, \quad b = -0.7327$$

Post-crash 88/05/18-94/11/25:

$$\sigma_{t+14} = a\sigma_t: \quad r^2 = .82, \quad a = .9899$$

$$\sigma_{t+14} - \sigma_t = (b/n)\log(S_{t+14}/S_t): \quad r^2 = .49, \quad b = -4.0715$$

$$\sigma_{t+14} = a\sigma_t + (b/n)\log(S_{t+14}/S_t): \quad r^2 = .91, \quad a = 1.0007, \quad b = -4.0800$$

Recovery of the stochastic process through implied binomial trees, as we have seen, strongly suggests that at-the-money implied volatility should vary inversely with the underlying asset price. We can make a quick check of this prediction.

The attached picture shows the results of regressions which attempt to use the current at-the-money volatility σ_t and the log return over the next 14 trading days $\log(S_{t+14}/S_t)$ to predict the at-the-money volatility 14 trading days into the future σ_{t+14} . In general, the option used to calculate σ_t and the option used to calculate σ_{t+14} will not be the same since the option that is at-the-money after 14 days will generally change since the underlying asset price has changed. The time period for the regressions, April 2, 1986 (the first day the S&P 500 Index calls traded as European options) to November 11, 1994, is divided into two subperiods, pre-crash (April 2, 1986 to September 4, 1987) and post-crash (May 18, 1988 to November 11, 1994).

The regressions over the pre-crash period show that adding the 14-day return to the current volatility does little to improve the prediction of the future volatility. This fits with what we already know about this period. Index option smiles were almost flat suggesting that the Black-Scholes formula, based on a constant volatility, worked well during this period. However, over the post-crash period, the 14-day return variable improves the prediction considerably.

In both cases, the coefficient a in the regression $\sigma = a\sigma_{t+1} + \varepsilon$, being near one, indicates that σ_t by itself is an unbiased forecast of σ_{t+14} . Surely, this is to be expected. Interestingly, this independent variable did a much better job forecasting the 14-day ahead volatility in the post-crash period. A second series of regressions sees how much of the variance of the forecast error $(\sigma_{t+14} - \sigma_t)$ can be explained by $\log(S_{t+14}/S_t)$. Pre-crash, this variable was of little assistance in helping explain this error, while in stark contrast, post-crash, this variable was of considerable value, confirming the prediction of implied binomial trees. Indeed, post-crash, taken together σ_t and $\log(S_{t+14}/S_t)$ explain 91% of the variance in σ_{t+14} .

Jump movements, not contemplated by Black-Scholes, could also potentially explain the observed left-skewness of the risk-neutral probability distribution if it were supposed that downward jumps are much more likely than upward jumps. Since 1987 there has been some rough empirical evidence of this in the US stock market. However, these jumps would not also explain the observed negative relation between volatility and index levels.



Economic Causes of Negative Skewness (and higher downside implied volatility)

- **Leverage:** as asset price falls \Rightarrow market value debt-equity ratio rises \Rightarrow **volatility rises**
- **Correlation:** equities become more highly correlated in down markets \Rightarrow **volatility rises**
- **Wealth:** as market falls \Rightarrow wealth falls \Rightarrow investors become more risk averse \Rightarrow same news leads to greater reaction and trading \Rightarrow **volatility rises**
- **Risk:** as **volatility rises** \Rightarrow risk premium increases \Rightarrow market falls

How to separate these potential causes? compare the volatility behavior of individual stocks vs market index.

Jumps: market is more likely to jump down than up \Rightarrow explains negative skewness, but not negative correlation between implied volatility and index level

To recapitulate, so far we have identified a significant departure in post-crash S&P 500 Index option pricing from the Black-Scholes formula. We have shown that this translates into a left-skewed highly leptokurtic risk-neutral distribution for the future underlying asset price. Using implied binomial trees, this further translates into a stochastic process for which the salient departure from Brownian motion is the inverse relation of implied volatility and the underlying asset price. Finally, we have verified that as predicted, this inverse relation was only present marginally pre-crash but was much stronger post-crash.

Perhaps we should ask what might be the economic causes of this departure from Black-Scholes. We are aware of four explanations in the current literature:

Leverage effect: When stock prices fall, the firm's debt-to-equity ratio in market value terms tends to rise since its denominator falls faster than its numerator. If returns from assets remain the same, the increasing debt-to-equity ratio magnifies the influence of return from assets on stock returns, thereby increasing volatility. Thus, indirectly through automatic changes in the debt-to-equity ratio, a fall in stock prices causes an increase in stock volatility. Not only should this affect smiles of individual stock options, but since index returns are a convex combination of constituent stock returns, a similar smile effect should be observed from index options.

Correlation effect: Suppose that when stock index prices fall, or fall significantly, individual stock returns become more highly correlated. Some empirical evidence supports this. For example, in the 1987 stock market crash, most stock markets around the world fell together. If this occurs, then with the attendant reduced advantage from diversification, volatility will rise.

Wealth effect: Suppose that when stock index prices fall, investors become noticeably less wealthy and because of this more risk averse. So that when the same type of information hits the market, they respond by buying more or selling more than they would have with higher stock prices. In turn, this causes stock prices to be more sensitive to news and volatility increases.

Risk effect: This reverses the order of causality of the wealth effect. In this case, something exogenous happens to increase stock market risk. Because investors are risk averse, they demand a higher expected return to hold stock. Assuming unchanged expectations, this leads to a reduction of current stock market prices.

It may be that each of these effects has some truth. One way to disentangle them is to compare the smiles for individual equities with the smile for indexes. In the US, post-crash, the S&P 500 Index smile is far more pronounced than the smiles observed for its constituent equities. This suggests that the leverage effect may be quite weak, lending increased weight to the other three possibilities [however, Toft and Prucyk 1997].



Types of Comparisons of Option Prices

Volatility-free testing of alternative option pricing models.

Comparisons among the prices of otherwise identical options but:

- ① with different striking prices: explain the current smile
- ② with different times-to-expiration: predict shorter-term smile from concurrent longer-term smile
- ③ observed at different points in time: predict future conditional smile from current smile
- ④ with different underlying assets: index option smiles v individual option smiles

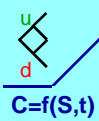
V. Empirical Tests of Alternative Forecasts of Risk-Neutral Probabilities

Although the implied binomial tree model correctly anticipates the negative relation between volatility and asset price, it is not the only option pricing model with this implication. This motivates our tests of alternative option pricing models. Earlier tests of option pricing models often relied on estimates of volatility from historically realized returns. Unfortunately, we cannot separate errors in volatility estimation from option formula errors, and it is very easy to err in volatility estimation. So in our tests, we will be careful to avoid such a joint hypothesis, and the tests will not depend on historical volatility estimates. Another problem with many tests of option pricing models is that they often rely on following the outcome of managing a sequentially revised portfolio (usually chosen to replicate an option payoff). This makes these results subject to questionable assumptions about transactions costs and errors in asset price measurement. Again, our tests will not rely on dynamic replication and so will also avoid these complications.

The attached picture contains four types of predictions from option pricing models that can be used, without relying on historical volatility estimation or dynamic replication, to test the validity of these models. The first and simplest simply compares the differential pricing of options across striking prices with model predictions [Rubinstein 1985]. Unfortunately, this simple test cannot be used for implied binomial trees because it takes these option prices as data and fits the stochastic process to them.

Another test is to compare the concurrent prices of otherwise identical options, but with different times-to-expiration. This can be used to test implied binomial trees (but not the generalized version). One can construct an implied tree from long-term options. Then this tree can be used to value options which mature earlier.

A more interesting and stronger test, and the one we will report here, is to construct a tree based on current option prices. And then to use this tree to predict the future prices of these same options. Having constructed a tree, as the future evolves, we can think of ourselves as moving along a path in the tree. If we stop before the options expire at the then current node, we can now infer a subtree from the original tree that should govern the stochastic movement of the underlying asset from that node forward to the expiration date. Using this inferred subtree, we can then value the options at that node and compare these to the market prices observed at that time. Stated equivalently, we can use the subtree to calculate the predicted implied volatilities of each option (the smile) and compare these to the observed smile in the market.



Empirical Test: Inferring Future Conditional Smiles from Current Option Prices

K/S	IMPLIED VOLATILITIES		
	159-day options in 30 days	91-day options now	189-day options now
.8748	?	(.2492)	.2278
.8892	?	(.2403)	.2209
.9035	?	(.2294)	.2139
.9179	?	(.2192)	.2049
.9322	?	(.2087)	.1961
.9466	?	(.1953)	.1888
.9609	?	(.1840)	.1814
.9752	?	(.1720)	.1718
.9896	?	(.1607)	.1645
1.0039	?	(.1506)	.1586
1.0183	?	(.1398)	.1503
1.0326	?	(.1298)	.1436
1.0756	?	(.1070)	.1244

Index
= 1.024

Index
= 1.000

A final test we have not yet performed is to compare smiles across different underlying assets, including a comparison of smiles for individual equity options with smiles for indexes.


We shall be comparing alternative option pricing models. The empirical test we shall emphasize uses the current prices of options to parameterize the models. Then the parameterized model is used to predict future option smiles. We then compare the predicted future smile with the actual smile subsequently observed in the market, conditional on knowing the new underlying asset price. We will prefer the model with a predicted smile which is closest (in absolute difference) to the realized smile.

The attached picture illustrates this test concretely. It lists options maturing in 189 days which have a striking price to current asset price ratios ranging from .8748 (out-of-the money puts) to 1.0756 (out-of-the-money calls). For example, the option with striking price to current asset price ratio of 1.0039 currently has an implied volatility of .1586. 30 days from now the options will have 159 days to expiration. At that time the underlying index rose from 1.000 to 1.024 (up 2.4%). Each option pricing model will supply a different prediction about the implied volatility of that option at that time. Our task will be to compare those predictions.

Although not reported in detail here, the alternative option pricing models were also parameterized using current prices for options expiring in 189 days, as well as current prices for options maturing in 91 days. Again the parameterized models are then used to predict option smiles in 30 days. We have not reported these results because they were little changed from the results where only options maturing in 189 days were used.



Alternative Option Pricing Models

- 1 **Black-Scholes (“flat smile”)** $[(dS)/S = \mu dt + \sigma dz]$
future implied σ s set equal to current at-the-money implied σ
- 2 **Relative Smile**
future implied σ s set equal to current implied σ s of options with same K/S
- 3 **Absolute Smile**
future implied σ s set equal to current implied σ s of options with same K
- 4 **Constant Elasticity of Variance: restricted** $[(dS)/S = \mu dt + \sigma' S^{\rho-1} dz]$
future (σ', ρ) set equal to best fitting current (σ', ρ) ($0 \leq \rho \leq 1$)
- 5 **Constant Elasticity of Variance: unrestricted**
future (σ', ρ) set equal to best fitting current (σ', ρ) ($\rho \leq 1$)
- 6 **Implied Binomial Trees** 
future option prices derived from implied binomial tree fitting current option prices
- 7 **Displaced Diffusion** $[S_t = (\alpha e^{\nu} + (1-\alpha)r^t)S_0]$
future (σ, α) [= % in risky asset] set equal to best fitting current (σ, α)
- 8 **Jump Diffusion** $[(dS)/S = (\alpha - \lambda k)dt + \sigma dz + dq]$
future (σ, λ, k) set equal to best fitting current (σ, λ, k)
- 9 **Stochastic Volatility** $[(dS)/S = \mu dt + v(t)^{1/2} dz_1, dv(t) = \kappa(\theta - v(t)) + v(t)^{1/2} dz_2]$
future $(\sigma, v(0), \kappa, \theta, \rho)$ set equal to best fitting current $(\sigma, v(0), \kappa, \theta, \rho)$

We have compared nine alternative approaches to option pricing. They can be grouped into four categories:

Standard benchmark model:

- n Black-Scholes model

“Naïve trader” models:

- n Relative smile model
- n Absolute smile model

Models emphasizing a functional relation between volatility and asset price:

- n Constant elasticity of variance diffusion: restricted
- n Constant elasticity of variance diffusion: unrestricted
- n Implied binomial trees
- n Displaced diffusion model

Models emphasizing other deviations from Black-Scholes:

- n Jump diffusion
- n Stochastic volatility

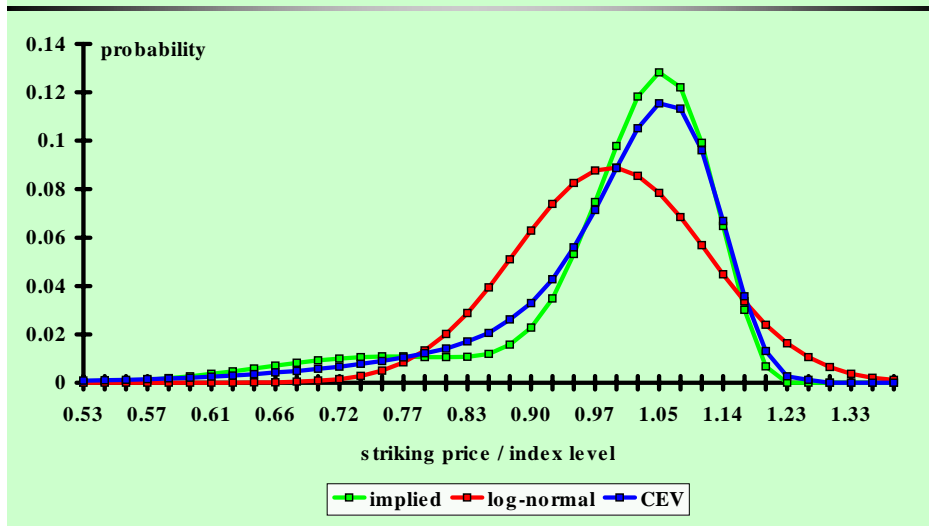
The Black-Scholes model is parameterized by setting the volatility parameter in the formula equal to the current at-the-money implied volatility. The prediction of the Black-Scholes model is that in the future all options will have that same implied volatility.

The “naïve trader” models are so named because they are simple rules of thumb commonly used by professionals. The relative smile model predicts that the future implied volatility of an option with striking price K when the underlying asset price is S_1 is the same as the current implied volatility of an option with a striking price equal to $K(S_0/S_1)$. In contrast, the absolute smile model predicts that the future implied volatility of an option with striking price K is the same as the current implied volatility of that option. For this model, it is as if for each option, its current implied volatility stays pinned to it.

The CEV restricted model assumes that the local volatility of the underlying asset is $\sigma' S^{\rho-1}$ where $0 \leq \rho \leq 1$ and σ' are constants [Cox 1996]. This model builds in directly an inverse relation between the local volatility and the underlying asset price S . The closer ρ is 0, the stronger this relation; and as ρ gets close to 1, the model becomes identical to the Black-Scholes formula. A more general version of this model which allows for an even stronger inverse relation we call the unrestricted version since it only requires that $\rho \leq 1$.



Lognormal, Implied and CEV Probability Distributions

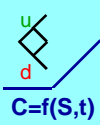


The displaced diffusion model is also based on the assumption that the volatility is a function of the underlying asset price [Rubinstein 1983]. As it was originally developed for individual stock options, the source of this dependence arose from the risk composition of the firm's assets and its financial leverage. Indeed, in contrast the CEV model, the displaced diffusion model actually permits the volatility to vary in the same direction as the underlying asset price if the asset composition effect is stronger than the leverage effect. But, I think we can anticipate in advance, that given the observed empirical inverse relation between asset price and volatility for both individual stocks and the index, the displaced diffusion model is likely to have no advantage over the CEV model in forecasting future implied volatilities (post-crash).

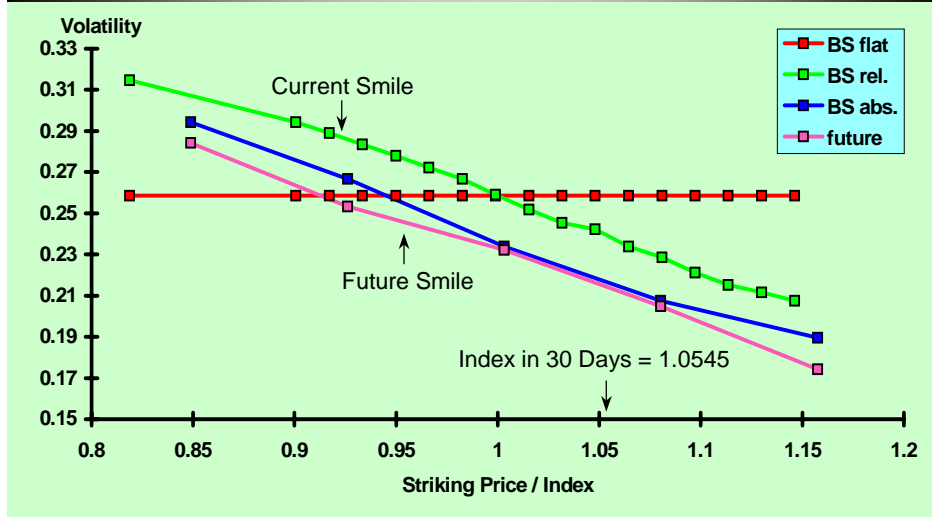
Many academics and professionals believe that diffusion-based option models which only allow the volatility to depend at most on the underlying asset price and time are too restrictive. So we also want to test models incorporating the two key other generalizations of the Black-Scholes formula: jump asset price movements and volatility which can depend on other variables. So we have included Merton's jump-diffusion model [Merton 1976] and Heston's stochastic volatility model [Heston 1993].

The attached graph shows illustrates the potential difference in estimated risk-neutral probability distributions of three of the alternative pricing models. It shows, as we have seen before, that the implied distribution is left skewed with much greater leptokurtosis than the lognormal. Notice that the CEV unrestricted model with a sufficiently low ρ parameter (about -4) fits the implied distribution reasonably well. However, the methods for inferring the implied stochastic process are different. In the CEV case, the ρ and σ' parameters determining the above fit are held fixed when the CEV formula is reapplied to value options in the future. The only changed inputs in the formula are S and t . Whereas, in the implied tree approach, a more elaborate backwards recursive tree construction is used, followed by the inference of the future subtree. In particular, in contrast to the CEV model, the implied tree approach builds in a dependence of the local volatility not only on the underlying asset price, but on time as well.

Nonetheless, because of the similarity between the two risk-neutral expiration-date distributions and because, as it turns out, the time-dependence of volatility appears slight, we can anticipate that the unrestricted CEV model will give similar results to the implied tree model in forecasting future implied volatility.



Comparative Naive Black-Scholes Predictions



Recall that we will be comparing the forecasts of future option implied volatility. Alternatively, we will be using alternative option pricing models, parameterized using current data, to forecast the prices of currently existing options at a specified future date before their expiration. That forecast will be conditional on knowing the underlying asset price at that future date since we are clearly not trying to forecast that as well. Having forecasted the future option prices, using the Black-Scholes formula, we will translate those prices into the metric of implied volatilities, construct the implied volatility smile, and compare these predicted implied smiles across the different pricing models.

The attached picture provides an illustration. The upper green sloped line is the current smile, summarizing along with the current underlying asset price (indexed to 1.00), most of the information we need to make our predictions. Note that it is considerably downward sloping, typifying the smiles for S&P 500 Index options in the post-crash period. The lower pink sloped line is the observed smile that was later observed at a specified future date when the index had risen to 1.0545. Not surprisingly, it too is downward sloping since we remain within the post-crash period. The prediction of the Black-Scholes model, based on constant volatility for all striking prices and time, is described by the horizontal red line. It simply says that the current at-the-money volatility of about 26% should continue to reign in the future for all the options.

The other two lines illustrate the predictions from our two “naïve trader” models. Since the horizontal axis is the ratio of the striking price to the underlying asset price, the relative smile model simply makes the prediction that the smile, scaled in terms of this ratio, will remain unchanged. So the green sloped line is at once the current smile and the prediction of the relative model of the future smile. In contrast, the absolute smile model predicts that options with the same striking price will have the same implied volatilities in the future that they have now. In the example, since the index moved up to 1.0545 in the future, the option that currently is at-the-money when the index is 1 will have in the future a striking price to index ratio of about .95. Since the option’s implied volatility is currently about 26%, the model predicts it will continue to have that same implied volatility in the future. Thus, the future prediction is graphed by the ordered pair (.95, .26) which is indeed a point along the lower blue sloped line containing the absolute model’s prediction. In general, if the index increases, the absolute model predicts that the smile will fall; while, if instead the index had fallen, the model would predict that the smile will rise.

Comparing the three predictions -- Black-Scholes, relative and absolute -- it is easy to see in the attached graph that the absolute model has worked best, since the blue line is closest to the pink line.



FUTURE V CURRENT pre-crash pricing errors

S&P 500 Index Options -- April 2, 1986 - October 16, 1987 (365 obs)

Error in Cents of 10(30)-Day Forecast

Forecasting Method	mean	median	std. dev.
Black-Scholes	50	39 (73)	36
Relative Smile	51	42 (72)	34
Absolute Smile	52	42 (74)	35
CEV: restricted	49	40 (72)	34
CEV: unrestricted	50	40 (72)	34
Implied Binomial Trees	54	44 (69)	40
Displaced Diffusion	50	40 (72)	34
Jump Diffusion	49	39 (71)	34
Stochastic Volatility	50	41 (72)	34

All models about the same.

For the pre-crash period, sampling once per day, the attached table summarizes the average absolute errors between the realized future smile and the smile prediction from each model. For each model two smile predictions are made, one 10 trading days in advance and the other 30 trading days in advance. For example, for the 10-day prediction, the Black-Scholes formula makes an average error of about 50 cents, and the median error across all the trading days is 39 cents. The median error for the 30-day prediction is about twice this at 73 cents.

All the models perform about the same. But this is just what would have been expected since all models nest the Black-Scholes formula as a special case and, as far as we can judge, the Black-Scholes formula worked quite well in this period. Even the relative and absolute are special cases of Black-Scholes since if the current smile were flat, both the relative and absolute models would predict that the future smile would remain unchanged.



FUTURE V CURRENT post-crash pricing errors

S&P 500 Index Options -- June 1, 1988 - December 31, 1994 (1562 obs)

Forecasting Method	Error in Cents of 10(30)-Day Forecast		
	mean	median	std. dev.
Black-Scholes	175	172 (171)	58
Relative Smile	73	55 (78)	61
Absolute Smile	56	44 (63)	43
CEV: restricted	139	136 (145)	54
CEV: unrestricted	74	56 (78)	60
Implied Binomial Trees	83	67 (86)	62
Displaced Diffusion	107	96 (112)	53
Jump Diffusion	168	164 (166)	56
Stochastic Volatility	75	57 (83)	67

Black-Scholes very poor @ 172¢. Absolute Smile best @ 44¢, but Relative Smile, CEV unrestricted, Implied Binomial Trees and Stochastic Volatility not far behind.

For the post-crash period, as expected the Black-Scholes model works very poorly, with a median absolute error of \$1.72 over a 10-day forecast period. The jump-diffusion model does almost as poorly. Again, given a strongly downward sloping smile, with the near-symmetric jump, up or down, of that model, we would not expect that it would offer much improvement. Smile patterns where the jump-diffusion model would help are weak smiles which turn up on both ends. Similarly, although the restricted CEV model can explain a downward sloping smile, it can only explain a much weaker slope, so it also offers little improvement.

However, substantial improvement over Black-Scholes is offered by the relative smile model, the absolute smile model, the unrestricted CEV model, implied binomial trees, and Heston's stochastic volatility model. Of these, the best performing is the absolute model. It is ironic that the simplest predictive rule (apart from Black-Scholes) does the best: every option simply retains whatever Black-Scholes implied volatility it started with. This model is a considerable improvement over Black-Scholes, reducing the median 10-day error to \$0.44, about one-fourth of the Black-Scholes error. The absolute smile model is also best over the longer 30-day prediction interval.*

I don't think we should conclude from this that academic attempts to improve the Black-Scholes model -- such as the CEV model, implied binomial trees, or the stochastic volatility model -- have therefore failed. Rather they do provide worthwhile improvements, cutting the Black-Scholes error to about one-third. But it is true that a "naïve trader" approach like the absolute smile model, which has no academic foundations, does even better. This throws down a challenge to academic and professional theorists to explain why the absolute model should work so well.

* The working paper by Dumas, Fleming and Whaley (1997) seems to contain a similar result. However, here, rather than emphasize the failure of implied binomial trees, we instead emphasize that implied binomial trees do much better than Black-Scholes and about as well as any competing "academic" model we have tested.



FUTURE V CURRENT: Summary

- **Best prediction is Absolute Smile. But, Relative Smile, CEV unrestricted, Implied Binomial Trees and Stochastic Volatility are not far behind.**
- **Post-crash 10-day pricing errors for these models are about 1/3 to 1/4 of Black-Scholes or CEV restricted errors.**
- **In general, knowing current short-term option prices in addition to long-term option prices doesn't seem to help.**
- **For Absolute Smile, post-crash 10-day forecast errors based just on the current long-term option prices are 44¢. This is cut to 23¢ if in addition the future ATM option price is assumed known. This error is further cut to 14¢ if in addition, errors are only measured outside the bid-ask spread.**

Our fascination with the absolute smile model led us to decompose its remaining \$0.44 error. We divided that error into three parts:

- n the error in predicting the future at-the-money volatility;
- n the error in predicting the implied volatility of other options, conditional on knowing the future at-the-money volatility;
- n the error if in addition it is assumed that transactions can only take place at the bid-ask prices, rather than at their midpoint.

Knowing the 10-day ahead at-the-money volatility in advance cuts the forecast error from \$0.44 to \$0.23 cents, or even further to \$0.14 if the error is measured relative to the bid-ask spread. This suggests that one way to approach future research on this issue is first to explain the changes in at-the-money volatility since that alone can explain about half of the \$0.44 error.



Naive Forecast of Future Implied Volatility

σ_t \equiv current ATM implied volatility

σ_{t+14} \equiv future implied volatility of same option (14 days later)

S&P 500 Index Options

Pre-crash 86/04/02-87/09/04:

$$\sigma_{t+14} = a\sigma_t: \quad r^2 = .49, \quad a = 1.0018$$

$$\sigma_{t+14} - \sigma_t = (b/n)\log(S_{t+14}/S_t): \quad r^2 = .02, \quad b = -.4268$$

$$\sigma_{t+14} = a\sigma_t + (b/n)\log(S_{t+14}/S_t): \quad r^2 = .50, \quad a = 1.0040, \quad b = -0.4813$$

Post-crash 88/05/18-94/11/25:

$$\sigma_{t+14} = a\sigma_t: \quad r^2 = .91, \quad a = .9709 \text{ ⚡}$$

$$\sigma_{t+14} - \sigma_t = (b/n)\log(S_{t+14}/S_t): \quad r^2 = .05, \quad b = -1.1705$$

$$\sigma_{t+14} = a\sigma_t + (b/n)\log(S_{t+14}/S_t): \quad r^2 = .92, \quad a = .9727, \quad b = -.8960 \text{ ⚡}$$

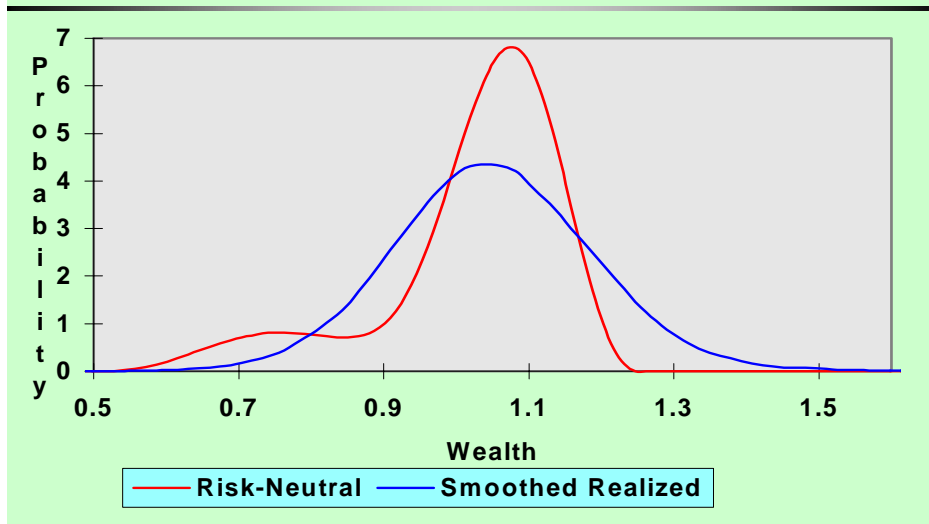
The success of the absolute model over the fancier academic models including implied binomial trees motivated us to test it directly in a time-series analysis. In our previous time series analysis, we compared the implied volatilities of options which were at-the-money at the beginning and at the end of a 14-day trading interval. The attached time-series results compare the implied volatilities of the same options at the beginning and end of 14-day trading intervals.

In the pre-crash period, using the implied volatility at the beginning of the period explains about half of the variance in the implied volatility of the same option at the end of the period, and with a coefficient close to one, provides an almost unbiased forecast. Adding the 14-day logarithmic return does little to improve this forecast. Again, given how well the Black-Scholes model fits option prices during this period, this should come as no surprise.

In the post-crash period, the beginning implied volatility now explains a much greater percentage of the variance of the ending volatility (91%) and continues to be a nearly unbiased forecast. When we looked at at-the-money implied volatility comparisons previously, we found that adding the log 14-day return substantially improved the forecast in the post-crash period. But, if the regressions are recast in terms of predicting 14-day ahead volatilities of the same options, then adding the log 14-day return offers almost no improvement in the forecast. This result is, of course, to be expected from our earlier analysis of comparative option pricing models.



Risk-Neutral v Realized Distributions (March 16, 1990)



VI. Recovering Risk Aversion

What kind of a market would produce risk-neutral distributions so much at variance with the Black-Scholes predictions?

- n One possibility is that post-crash the market dramatically changed the subjective probabilities it attached to the future performance of the S&P 500 index.
- n Another possibility is that the market, post-crash, became much more averse to downside risk.

If, following a time-honored tradition in financial economics, we measure the consensus market subjective probability distribution by its future realized frequency distribution, the result is described by the blue (almost normal) line in the attached graph. Superimposed in the red line is the risk-neutral distribution deduced by our techniques from March 15, 1990 S&P 500 option prices.

The difference between the two distributions is striking. If we have measured the market subjective distribution accurately, then the shape of this distribution has not changed very much pre- and post-crash. So we must look elsewhere for an explanation of the post-crash risk-neutral distribution, perhaps to changed risk-aversion.

But before taking a look at this, we need to discuss an important objection. Using the realized frequency distribution, either drawn from realized prices prior to March 15, 1990 or from realized prices after March 15, 1990, time-honored though it may be, is a highly suspect measure of the subjective distribution that was actually in the minds of investors. In particular, if the market were anticipating an improbable but extreme event (such as a second crash) which had not yet been realized, it would not show up in our estimate of the subjective distribution. At the same time, these events may be very important, despite their infrequency, to understanding the pricing of options, particularly out-of-the-money puts.

Our way around this problem is to draw implications for market-wide risk aversion only from the comparative shapes of the realized and risk-neutral distributions around their means, without needing to consider the more questionable tails of these distributions. Around the means, it seems likely that the realized distribution provides a reasonably reliable approximation of the true subjective distribution in this region. In addition, our earlier analysis also shows that our techniques for estimating risk-neutral distributions from option prices are very robust around the means to alternative methods since available options are dense in this region.



Notation

$P_j \equiv$ **risk-neutral probability** for state $j = 1, \dots, n$

$Q_j \equiv$ **subjective probability** for state j

$R_j \equiv$ **market portfolio return (ex-payout)** for state j

$\delta \equiv$ **market portfolio payout return over a single period**

$r \equiv$ **riskless return over a single period**

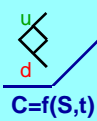
$U(R_j \delta^n) \equiv$ **utility function of representative investor**

$$\text{Max}_{R_j} \sum_j Q_j U(R_j \delta^n) - \lambda [(\sum_j P_j R_j) / (r/\delta)^n - 1]$$

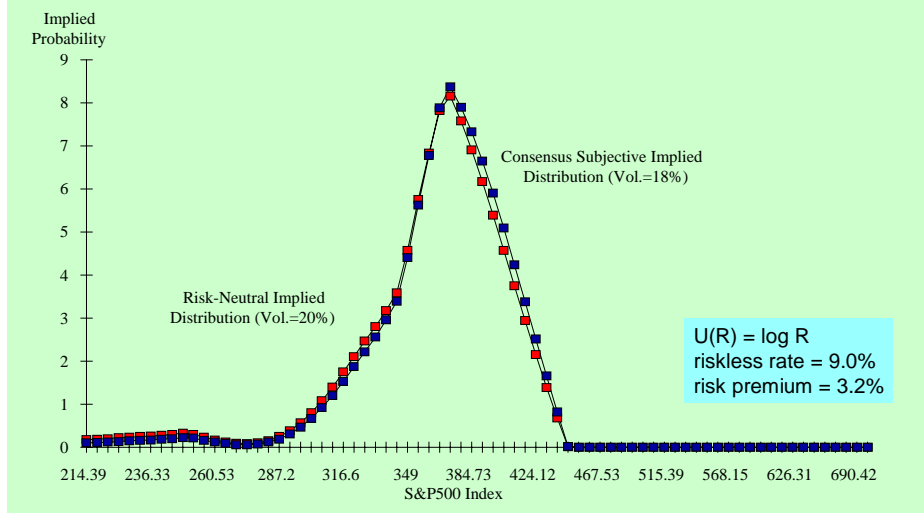
$$\text{differentiating once: } U'(R_j \delta^n) = \lambda (P_j / Q_j) / r^n$$

Given the risk-neutral distribution, we can estimate the subjective distribution by imposing popular risk aversion assumptions. We infer risk-aversion using a simple but widely-used model of financial equilibrium. The attached picture defines the variables we will be using. Assuming a consensus investor, we maximize his expected utility $\sum_j Q_j U(R_j \delta^n)$ subject to a constraint anchoring his present wealth $(\sum_j P_j R_j) / (r/\delta)^n$ to 1. Choosing his portfolio of state-contingent claims is equivalent to choosing the returns R_j he will realize in each state j . [δ^n is a correction for R_j which is defined only to be the market portfolio return after payouts, so that $R_j \delta^n$ is the market portfolio's total return.]

At the optimal choice, we have the familiar first-order condition: $U'(R_j \delta^n) = \lambda (P_j / Q_j) / r^n$. Except for λ , this is a state-by-state restriction on the relation of risk-aversion, subjective probabilities and risk-neutral probabilities



Subjective and Risk-Neutral 164-Day Probabilities S&P500 Index Options: January 2, 1990 (11:00 am)



If we assume logarithmic utility so that $U(R_j \delta^n) = \log(R_j \delta^n)$, then this first order condition becomes:

$$1/(R_j \delta^n) = \lambda(P_j/Q_j)/r^n$$

so that

$$Q_j = \lambda P_j R_j (\delta/r)^n$$

Summing over all j and since $\sum_j Q_j = 1$:

$$1 = \lambda (\sum_j P_j R_j) (\delta/r)^n$$

Since the investor is constrained so that $\sum_j P_j R_j (\delta/r)^n = 1$, then $\lambda = 1$.

Substituting this into one of the above equations, leads to the very simple decomposition of subjective probabilities:

$$Q_j = (P_j/r^n)(R_j \delta^n)$$

so that the subjective probability of a state equals the state-contingent price for that state weighted by the total market return in that state.

The attached graph shows the relation of subjective and risk-neutral probabilities for Jan 2, 1990 if we derive the subjective probability distribution, not from past or future index realizations, but from a simple model of financial equilibrium based on logarithmic utility and risk-neutral probabilities estimated from option prices.

Note how close the risk-neutral and subjective distributions are. The main difference is that the subjective distribution is shifted to the right with a mean of 12.2% in contrast to the 9% mean of the risk-neutral distribution. The risk aversion property of logarithmic utility accounts for this shift. But the shapes of the two distributions are almost the same.

This contrasts sharply with our previous comparison of subjective and risk-neutral distributions, where the risk-neutral distribution was estimated in the same way, but the subjective distribution was estimated from realized index prices. Clearly, a simple model of equilibrium with logarithmic utility does not explain this disparity.

So we now ask what consensus utility function could simultaneously rationalize these two distributions.



Equilibrium Preference-Probability Relation

$$\text{Max}_{R_j} \sum_j Q_j U(R_j \delta^n) - \lambda [(\sum_j P_j R_j)/(r\delta)^n - 1]$$

$$\text{differentiating once: } U'(R_j \delta^n) = \lambda (P_j/Q_j)/r^n$$

$$\text{differentiating twice: } U''(R_j \delta^n) = (\lambda \delta^n r^n) [(P_j' Q_j - P_j Q_j')/Q_j^2]$$

$$\text{combining: } -U''(R_j \delta^n)/U'(R_j \delta^n) = (\delta^{-n}) [(Q_j'/Q_j) - (P_j'/P_j)]$$

This shows how absolute risk-aversion for a given state is related to **subjective** and **risk-neutral** probabilities for that state, independent of other states. With this, we can examine center states which have the highest probability while neglecting the notoriously unreliable tail estimates.

A trick to this comparison is to differentiate the general first-order condition a second time to obtain another condition that needs to hold in equilibrium:

$$-U''(R_j \delta^n)/U'(R_j \delta^n) = (\delta^{-n}) [(Q_j'/Q_j) - (P_j'/P_j)]$$

$Q_j' (P_j')$ is the change in the subjective (risk-neutral) distribution across the nearby state. For example $Q_j' = \partial Q_j / \partial S_j$ and is approximated by $(Q_{j+1} - Q_{j-1}) / (S_{j+1} - S_{j-1})$.

This has the advantage of being a true state-by-state condition, where λ has been eliminated. This permits us to examine only the states near the mean in which we have the greatest confidence of our estimate of the subjective distribution inferred from realizations. In particular, we can determine the utility function fit in this region without needing to estimate the shape of the tail probabilities in which we have very little confidence. This condition also conveniently isolates the measure of absolute risk aversion on its left-hand side.



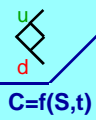
Methodology

- **Risk-neutral probability distributions:**
 - inferred from S&P500 index options with 135-225 days-to-expiration
 - using the maximum smoothness method
- **Other parameters (S , r , d , t) as observed in the market**
- **Subjective probability distributions:**
 - bootstrapped from 4-year historical samples
 - 25,000 returns matching the options time-to-expiration are generated and smoothed through a Gaussian kernel
 - mean is reset to riskfree rate plus 7% annualized
 - volatility is reset to volatility of risk-neutral distribution

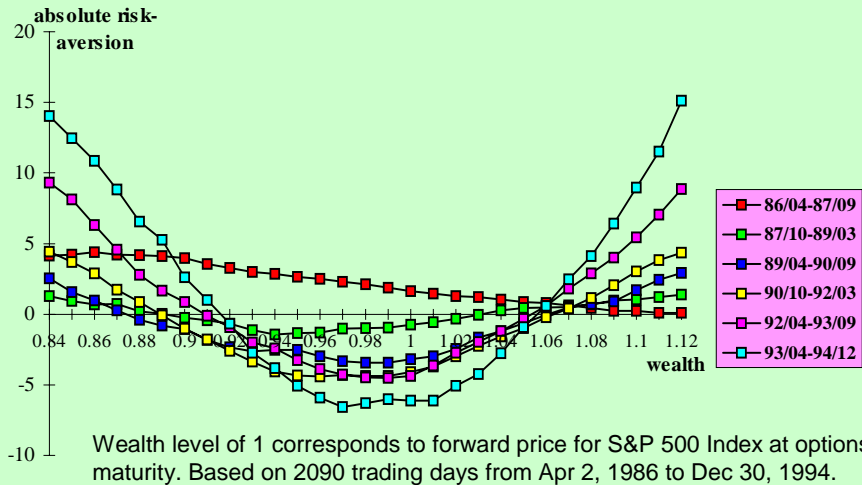
With our equilibrium result for absolute risk aversion in hand, the risk-neutral and subjective distributions were estimated following the techniques described in the attached picture for several non-overlapping time periods from April 2, 1986 to December 30, 1994.*

Unreported tests show that the estimated subjective probabilities are robust to perturbations in all of these assumptions. In particular, assuming a risk premium in the range of 5%-10% leaves the results essentially unchanged.

 *The use of a bootstrapping method destroys serial correlation. However, work in progress indicates that a lognormal distribution (which is the result of destroying serial dependence) provides a reasonable fit to these half-year returns [Jackwerth, February 1997]. On another matter, the risk-neutral distribution is strictly a point estimate. Some results concerning the degree to which probabilities can vary around point estimates are contained elsewhere [Jackwerth, March 1997].



Absolute Risk Aversion Across Wealth



The resulting absolute risk aversion is described by the attached graph for each time period. For example, for the single pre-crash period, April 1986 to September 1987, absolute risk aversion is positive but more or less declining with increasing wealth, and within the range 0 to 5 - a plausible result. Unfortunately, in all the post-crash periods the results make no sense. Absolute risk aversion is not only increasing over levels of wealth greater than current wealth, but is even negative over the range .9 to 1.06 times current wealth. What is even more, this bizarre result worsens as we move further and further into the future from the 1987 crash.

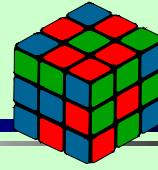
This result is essentially being driven by the extreme difference between the risk-neutral and measured subjective distribution around the mean. As we saw in an earlier graph [Figure 25], post-crash, on both sides of the mean, the risk-neutral distributions changed much more rapidly than the subjective distribution. It is simply the case that our equilibrium model cannot make sense of this

For the first order condition of this model to be a necessary condition of equilibrium, the second order condition which requires a negative second derivative of the utility function must hold. But for absolute risk aversion to be negative, either $U' < 0$ or $U'' > 0$. If, for example, $U' > 0$ but $U'' > 0$, then the first order condition need not characterize the optimum. In essence, the attached picture says that something is seriously wrong somewhere.

If the assumed risk premium is pushed from 7% to as high as 23%, the lines of absolute risk aversion now all fall just above the horizontal axis, so that $U' > 0$ and $U'' < 0$. However, even in this extreme case, the shape of the lines remains about the same. In particular, post-crash, they continue to exhibit increasing absolute risk aversion in the range above current wealth.



Potential Explanations



- Representative investor is a poor assumption.
- S&P 500 is a poor proxy for the market portfolio.
- Utility functions depend on other variables besides wealth.
- More general frameworks for utility functions admitting risk preference (prospect theory).
- The subjective distribution Q is not well-approximated by realizations.
- Trading costs, particularly for deep-out-of-the-money put options.
- Mispricing of deep-out-of-the-money puts and calls.

So what could be wrong? Something we have assumed must be at fault. One possibility is that our use of a representative investor could be a very bad assumption. Or the S&P 500 could be a very poor approximation of the market portfolio. For that to be true, the market portfolio must be relatively uncorrelated with the returns of the S&P 500 Index. Utility could be a function of other significant variables besides wealth. Perhaps investors prefer risk over a range of their future wealth, such as suggested by prospect theory.

Even though our results only depend on probabilities near the mean, it may still be the case that historically realized returns are not reliable indicators of subjective probabilities

So far, for the most part, we have ignored trading costs. In particular, although some methods used for estimating risk-neutral probabilities require that securities valued under these probabilities fall within the bid-ask spread, even these methods do not give full consideration to the role of trading costs in all its varying guises - commissions, bid-ask spread and market impact. For example, it may be that the relatively high prices of out-of-the-money puts which drive the post-crash S&P 500 Index smile, are somehow the result of trading costs that we have not considered. To us, given the magnitude of the smile effect and the high absolute dollar prices of these options (since the underlying asset is scaled to a high price), this seems unlikely but it should not be dismissed without a deeper analysis.

Another problem with looking to trading costs as the solution to the puzzle is that the implied risk-neutral distribution changed markedly before to after the stock market crash, yet it seems unlikely that trading costs did.

Finally, although it may be heretical to suggest this, the high prices of out-of-the-money puts may be the result of mispricing that a normally efficient market fails to correct. For some reason, enough capital may not be mobilized to sell these types of options.



Adjusted Excess Return Measure

- Assume the market portfolio exhibits lognormal returns
- Instead of β , we use $B \equiv \text{COV}[r_p, -r_m^{-b}] / \text{COV}[r_m, -r_m^{-b}]$, where B is an adjusted beta measure for the portfolio
 - $r_p \equiv$ return of the portfolio
 - $r_m \equiv$ return on the market
 - $r \equiv$ riskless return
- In this case: $b = (\ln(E[r_m]) - \ln r) / \text{Var}[\ln r_m]$
- Instead of α , we use $A \equiv (r_p - r) - B(r_m - r)$ where A is an adjusted portfolio expected excess return

We consider this last possibility by examining the returns from following a strategy where out-of-the-money six-month S&P 500 Index puts are sold every three months during the post-crash period. Each period we assume that the number of puts sold equals the number that could be sold with \$100 margin under the requirement that the margin for a sold uncovered put is 30% of the index level less the out-of-the-money amount. We compare these realized returns to risk, measured by a version of the capital asset pricing model which considers positive preference toward skewness, an aspect of investor preferences that may be important in the pricing securities with adjustable asymmetric outcomes such as options. [Rubinstein 1976, Leland 1996].

The attached picture states that replacing:

$$\beta \equiv \text{Cov}(r_p, r_m) / \text{Var } r_m$$

is the generalized risk measure (adjusted beta):

$$B \equiv \text{Cov}(r_p, r_m^{-b}) / \text{Cov}(r_m, r_m^{-b})$$

where r_p is the return (one plus the rate of return) of an arbitrary portfolio, r_m is the return of the market portfolio, and b is the consensus market relative risk aversion.

Using this measure, the realized excess over risk-adjusted return is:

$$A \equiv (r_p - r) - B(r_m - r)$$

which we call the realized adjusted alpha.

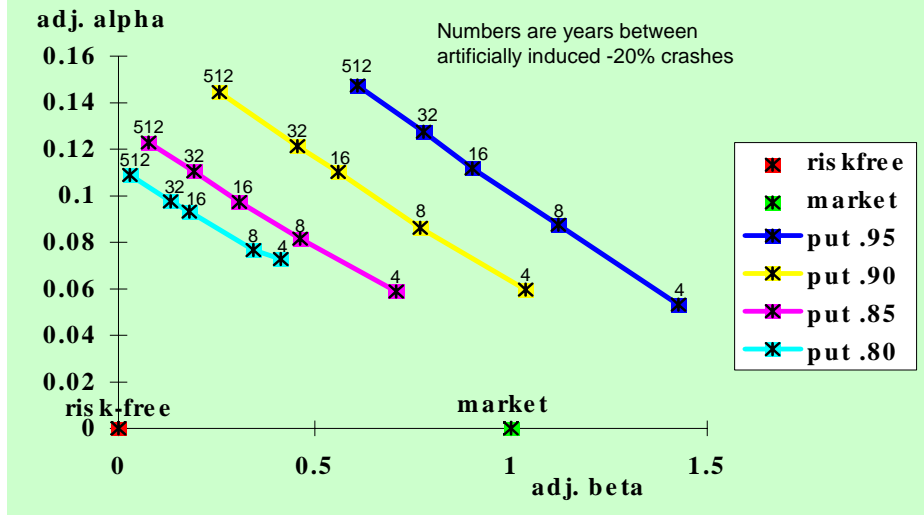
Based on the formula:

$$b = (\ln(E[r_m]) - \ln r) / \text{Var}[\ln r_m]$$

we set $b = 3.63$. But even if b were as high as 10, our results would be essentially unchanged.



Excess Returns Selling Out-of-the-Money Puts



The attached graph shows the results of our adjusted alpha-adjusted beta return analysis. The riskless return itself is located at the origin, and the market return is located along the horizontal axis at 1 (adjusted alpha of 0, adjusted beta of 1).

Each line looks at the alpha-beta ordered pairs for strategies using puts of varying degrees of being out-of-the-money. For example, for the upper blue line, the puts sold were about 5% out-of-the-money at the time of sale.

An important objection to our analysis as it has so far been described is that our strategy of selling out-of-the-money puts may do well in the post-crash periods because the much-feared second crash has not yet occurred, and had it occurred our strategies would have done poorly. To allow for this, we have inserted crashes into the data at varying frequencies. For example, the alpha-beta ordered pairs labeled 4 are constructed from the time-series of S&P 500 returns by adding crashes of the October 19, 1987 magnitude (down 20% in a single day) at the expected rate of once every four years. That is, each day a number is drawn at random with replacement from a bowl containing about 999 zeros and 1 one. If 0 is drawn, no crash occurs. If 1 is drawn, then the return for that day is adjusted downward by 20% and future returns continue as before unless yet another crash is drawn. For example, if the return for that day were actually -1%, we assume instead that the return was -21%

Thus, the exhibit shows that with almost no crashes added to the post-crash historical record (the ordered pairs labeled 512), the adjusted alpha ranges from 11% to 15% per annum. Thus, given the actual outcomes in the post-crash period, the strategy of selling out-of-the-money puts would have beaten the market by a good margin, and the more out-of-the-money the puts the better.

Perhaps this is not surprising since almost no artificially induced crashes have been added to the realized historical time series. However, at the other extreme, suppose a crash is inserted into the data with a frequency of every four years. This means that about two daily 20% crashes during the post-crash periods were assumed to occur. In that case, the adjusted alpha is about 6%-7% per annum. Even in this case, the strategy produces superior returns

Now you may object that our result does not adequately consider the extreme fear the market may have of downside returns. But we have given some consideration to this because we have been careful to adjust our risk measure for dislike of negatively skewed returns, for a relative risk aversion level of $b = 3.63$. Also imposition of commissions plus bid-ask spread transactions costs, with a crash expected every 4 years, still leaves adjusted alphas in the range of 4%-5% per annum.



Summary

Motivation: post crash failure of the Black-Scholes formula in the index options market

- **Recover risk-neutral probabilities from option prices**
 - robust methods available
 - leptokurtosis and left-skewness
- **Imply a consistent risk-neutral stochastic process**
 - simple generalization of standard binomial trees
- **Empirical tests of alternative smile forecasts**
 - “naïve trader” model best, several academic models similar
- **Recover risk aversion from realized returns and option**
 - strange results, index option market possibly inefficient



Bibliography (principal references)

- Rubinstein, M., "Implied Binomial Trees," *Journal of Finance* (July 1994).
- Jackwerth, J. and M. Rubinstein, "Recovering Probability Distributions from Option Prices," *Journal of Finance* (December 1996) [type ① comparison].
- Jackwerth, J., "Generalized Binomial Trees," *Journal of Derivatives* (Winter 1997) [type ① and ② comparisons].
- Jackwerth, J., "Recovering Risk Aversion from Option Prices and Realized Returns," *Review of Financial Studies* (Summer 2000) [type ① comparisons].
- Jackwerth, J.C. and M. Rubinstein, "Recovering Stochastic Processes from Option Prices," working paper in progress [type ② and ③ comparisons].

Bibliography

- Black, F. and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81, No. 3 (May-June 1973), pp. 637-659.
- Breeden, D.T. and R.H. Litzenberger, "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business* 51, No. 4 (October 1978), pp. 621-651.
- Cox, J.C., "The constant elasticity of variance option pricing model" *Journal of Portfolio Management* (Special Issue: A Tribute to Fischer Black, December 1996), pp. 15-17.
- Cox, J.C., S.A. Ross and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7, No. 3 (September 1979), pp. 229-263.
- Dumas, B., J. Fleming and R. Whaley, "Implied Volatility Functions: Empirical Tests," *Journal of Finance* 53, No. 6 (1998), pp. 2059-2106.
- Gemmill, G. and N. Kamiyama, "International Transmission of Option Volatility and Skewness: When You're Smiling, does the Whole World Smile?" City University Business School, London, working paper (February 1997)
- Heston, S.L. "A Closed-Form Solution for Options with Stochastic Volatility and Applications to Bond and Currency Options," *Review of Financial Studies* 6, No. 2 (Summer 1993), pp. 327-343.
- Jackwerth, J., "Generalized Binomial Trees," *Journal of Derivatives* 5, No. 2 (1997), pp. 7-17.
- Jackwerth, J., "Recovering Risk Aversion from Option Prices and Realized Returns," *Review of Financial Studies* 13, No. 2 (2000), 433-451.
- Jackwerth, J.C., "Do We Live in a Lognormal World?" University of Wisconsin at Madison, working paper in progress (February 1997).
- Jackwerth, J.C., "Geometric and Probabilistic Bounds on Option Prices," University of Wisconsin at Madison, working paper in progress (March 1997).
- Jackwerth, J.C. and M. Rubinstein, "Recovering Probabilities from Option Prices," *Journal of Finance* 51, No. 5 (December 1996), pp. 1611-1631.
- Jackwerth, J.C. and M. Rubinstein, "Recovering Stochastic Processes from Option Prices," University of Wisconsin at Madison and University of California at Berkeley, working paper in progress (2001).
- Leland, H., "Beyond Mean-Variance: Risk and Performance Measurement in a Nonsymmetrical World," *Financial Analysts Journal* 55, (1999), pp. 27-36.
- Merton, R.C., "Option Pricing When Underlying Stock Returns are Discontinuous," *Journal of Financial Economics* 3, No. 1 (January-March 1976), pp. 125-144.
- Rubinstein, M., "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics and Management Science* 7, No. 2 (Autumn 1976), pp. 407-425.
- Rubinstein, M., "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976 through August 31, 1978," *Journal of Finance* 40, No. 2 (June 1985), pp. 455-480.
- Rubinstein, M., "Displaced Diffusion Option Pricing," *Journal of Finance* 38, No. 1 (March 1983), pp. 213-217.
- Rubinstein, M., "Implied Binomial Trees," *Journal of Finance* 49, No. 3 (Presidential Address to the American Finance Association, July 1994), pp. 771-818.
- Toft, K.B. and B. Prucyk, "Options on Levered Equity: Theory of Empirical Tests," *Journal of Finance* 52, No. 3 (July 1997), pp. 1151-1180.